2. THE "STANDARD MODEL" OF NR QUANTUM THEORY

Consider 1 NR e^-, spin $\frac{1}{2}$.

$$\Gamma = T^*\mathbb{R}^3 \times S^2 \xrightarrow[\hbar]{} \mathcal{H}_e, \vec{p} = \frac{\hbar}{i}\vec{\nabla}, \dots$$

Hilbert space of state vectors

$$\mathcal{H}_e = L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$$

Spin (from S^2): $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$,

$\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$: Pauli matrices

Schrödinger-Pauli eq.:

$$i\hbar \frac{\partial}{\partial t}\psi_t = H_t \psi_t,$$

$\psi_t \in \mathcal{H}_e$, $H_t = H_t^*$ given by

$$H_t := \underbrace{\frac{1}{2m}\vec{\pi}^{\,2}}_{kE} - \underbrace{\frac{1}{2mc}\left\{\vec{\pi}\,;\left(\vec{\mu} - \frac{e}{2mc}\vec{S}\right)\wedge \vec{E}_t\right\}}_{S-O} - \underbrace{e\phi_t}_{pE} - \underbrace{\vec{\mu}\cdot\vec{B}_t}_{Z}\,, \qquad \text{where}$$

$$\vec{\pi} \equiv m\vec{V} := \frac{\hbar}{i}\vec{\nabla}_x - \frac{e}{c}\vec{A}(\underline{x},t)$$

$$\vec{\mu} := \frac{g\,\mu_{Bohr}}{\hbar}\vec{S}\,, \qquad \vec{S} = \frac{\hbar}{2}\vec{\sigma}\,,$$

$$g = 2\,, \quad \mu_{Bohr} = -\frac{e\hbar}{2mc} \simeq 5.79\times 10^{-9}\,\frac{eV}{Gauss}$$

$\vec{E}_t$: external electric field

$\vec{B}_t$: ext. magnetic field

$\vec{A}_t$: em vector potential

$U(1)_{em} \times SU(2)_{spin}$ gauge inv.

$$D_\mu := \frac{\partial}{\partial x^\mu} + i a_\mu + w_\mu \,, \quad \mu = 0,1,2,3$$

$$a_0 = \frac{e}{\hbar c}\phi \,, \quad \vec{a} = -\frac{e}{\hbar c}\vec{A} \,,$$

$$w_\mu = \sum_{A=1}^{3} w_{\mu A}\, i\sigma_A \,, \quad \text{with}$$

$$w_{0A} = -\frac{g\mu_{Bohr}}{2\hbar c} B_A + \cdots$$

$$w_{kA} = \left(-\frac{g\mu_{Bohr}}{2\hbar c} + \frac{e}{4mc^2}\right)\sum_{M=1}^{3} \varepsilon_{kAM} E_M + \cdots$$

Adding a term $\propto O(\vec{w}^{\,2})$, Pauli eq. becomes :

$$i\hbar c D_0 \psi_t = -\frac{\hbar^2}{2m} \vec{D}^2 \psi_t \quad (PE)$$

Form of this eq. <u>invariant</u> under $U(1) \times SU(2)$ gauge transformations:

$$a \longmapsto a + d\chi$$

$$w \longmapsto g w g^{-1} + g d g^{-1}$$

$$\psi \longmapsto e^{-i\chi} g \psi$$

$$\chi = \chi(\underline{x}, t) \in \mathbb{R}, \quad g = g(\underline{x}, t) \in SU(2)$$

ψ section of vector bundle assoc. to principal $U(1) \times SU(2)$-

bundle; base space $\mathbb{R}^3$.

$a \leftrightarrow U(1)$ connection A-B

$w \leftrightarrow SU(2)$ — " — A-C effects

General form of a, w

Equip $\mathbb{R}^3$ with curved metric; study Pauli eq. in moving frame det. by $\vec{V}(\underline{x},t)$, $\vec{\nabla}\cdot\vec{V} \equiv 0$.

$$a_0 = \frac{e}{\hbar c}\phi - \frac{m}{2}\vec{V}^2 - \frac{e}{\hbar c}\vec{V}\cdot\vec{A}$$

$$a_k = -\frac{e}{\hbar c}A_k - \frac{m}{\hbar}V_k$$

$$w_{0A} = -\left\{ \frac{g\mu}{2\hbar c} B_A - W_A + \left(-\frac{g\mu}{2\hbar c} + \frac{e}{4mc^2} \right) \frac{1}{c} (\vec{V} \wedge \vec{E})_A + \frac{1}{c} \Omega_A \right\} + (\propto \omega)$$

where

$\mu = \mu_{Bohr}$, $\vec{W}$: Weiss exchange field

$\vec{\Omega} = \frac{1}{2} \vec{\nabla} \wedge \vec{V}$: vorticity

$$w_{kA} = \cdots + \omega_{kA}$$

ω : spin conn. assoc. to ∇_{LC}

$w \longleftrightarrow$ Magnetism, high-T_c

Pauli eq. is Euler-Lagr. eq. derived from $U(1)\times SU(2)$ gauge-inv. action funct.

$$S(\psi^*,\psi;a,w) =$$

$$\int dt \int_{\mathbb{R}^3} d\,vol. \left\{ i\hbar c\, \psi^* D_0 \psi - \right.$$

$$\left. - \frac{\hbar^2}{2m} (\vec{D}\psi)^* \cdot \vec{D}\psi - U(\psi^*,\psi) \right\}$$

↑
ext. pot, 2-body int.

Starting point for funct. integral approach to many-body theory, incorp. Pauli Excl. Principle.

U(1)×SU(2) gauge invariance common to all systems of non-relat. qm particles w. spin and, w. PP, underlies following phys. phenomena:

U(1) gauge invariance

↔ A-B, spectroscopy – R & M
metals, el. conductivity
quantum Hall effect
London eq., supercon-ductivity, super-fluidity, …

SU(2) gauge invariance

$\leftrightarrow$

- Aharonov-Casher $\leftrightarrow \vec{E}$
- Zeeman effect $\leftrightarrow \vec{B}$
- Bloch spin res., Einstein-de Haas-Barnett effect $\leftrightarrow \vec{B}, \vec{\Omega}$
- magnetism $\leftrightarrow \vec{B}, \vec{W}$
- high-T_c superconductivity $\leftrightarrow \vec{W}$

QM Larmor theorem

System of electrons in ext. $\vec{B}$-field, vector pot. $\vec{A}$, $\vec{\nabla}\cdot\vec{A}=0$.

To 1^{st} order in $\vec{B}$, can eliminate $\vec{B}$ by passing to moving frame w. $\vec{V} = -\frac{e}{mc}\vec{A}$

$\stackrel{e.g.}{\Longrightarrow}$ • Einstein-deHaas-Barnett,

• London equation:

$$\vec{J}_s \equiv q n_s \vec{V} = -\frac{q^2 n_s}{Mc}\vec{A}$$

$$q = 2e\,, \quad M \simeq 2m\,.$$

• Quant. flux of vortices:

supercond. – superfluids

$$\Phi_{\vec{B}} = \frac{hc}{q} n \longleftrightarrow \Phi_{\vec{V}} = \frac{h}{M} n\,,$$

$$n \in \mathbb{Z}$$

$$q = 2e \qquad M \stackrel{e.g.}{=} m_{^4He}, 2m_{^3He}$$

3. PAULI's ELECTRON $\leftrightarrow$ SUSY QM $\leftrightarrow$ DIFF. GEOMETRY & -TOPOLOGY

Consider an e^- in $(\mathbb{R}^3, g)$.

$$\mathcal{H}_e = L^2(\mathbb{R}^3, dvol_g) \otimes \mathbb{C}^2.$$

Introduce Pauli-Dirac op.:

$$\mathcal{D} := \sum_{k=1}^{3} \sigma^k (-i\nabla_k)$$

∇: spin conn. on spinor bundle over $(\mathbb{R}^3, g)$ assoc. w LC

Neglect electrostat. potential, spin-orbit interactions & exchange field. Then

$$\nabla_k = \partial_k - i\frac{e}{\hbar c} A_k + \frac{1}{2}\omega_{k,ab}[\sigma^a;\sigma^b]$$

spin connection

Susy form of Pauli eq.:

$$i\hbar\,\partial_t \Psi_t = \frac{\hbar^2}{2m} \mathcal{D}^2 \Psi_t \,.$$

$\leftrightarrow$ $g=2$! inv. under time-indep. $U(1)\times SU(2)$ gauge trsfs.

Set
$$Q = \frac{\hbar}{\sqrt{2m}} \mathcal{D}$$

Q is "supercharge" of $N=1$ susy QM, with $H = Q^2$!

- Analogous theory for NR positrons: Pass to conjug. spinor bundle, $e \to -e$.
- Positronium = electron–positron groundstate

State space:

$$L^2(\mathbb{R}^3, d\mathrm{vol}_g) \otimes (\Lambda_+ \oplus \Lambda_-),$$

$$\Lambda_\pm := (\mathcal{A}_0 \oplus \mathcal{A}_1)_{\pi=\pm}$$

square-int. diff. forms on $\mathbb{R}^3$

$\longrightarrow$ $N=2$ susy QM, super-charges d, d^*,

$$H = (2M)^{-1}(dd^* + d^*d).$$

Generalize this theory:

$$(\mathbb{R}^3, g) \longrightarrow (M, g)$$

M: gen. Riem. (spin^c) mf.

Can recover all of diff. top. & diff. geom. from susy QM of "electron" & "positronium"; special geom. $\leftrightarrow$ higher susy.

2. Rediscovering diff. geometry & topology from QT of electron

Single electron in $(\mathbb{R}^3, g)$.

$$\mathcal{H}_e = L^2(\mathbb{R}^3, dvol_g) \otimes \mathbb{C}^2 .$$

Pauli-Dirac operator:

$$D := \sum_{k=1}^{3} \sigma^k (-i \nabla_k),$$

$$\nabla_k = \partial_k - i \frac{e}{\hbar c} A_k + \frac{1}{2} \omega_{k,ab} [\sigma^a, \sigma^b]$$

spin connection

Neglect spin-orbit interactions, exchange field $\vec{W}$.

"Susy" form of Pauli eq.

$$i\hbar\partial_t\psi_t = \frac{\hbar^2}{2m}D^2\psi_t - e\phi_t\psi_t$$

inv. under time-indep.
$U(1)\times SU(2)$ gauge trsfs.

Set $Q := \frac{\hbar}{\sqrt{2m}}D$, $\hbar = 1$.

Q : "supercharge" $(N = 1)$.

If $\phi_t \equiv 0$ then

$$i\partial_t\psi_t = Q^2\psi_t$$

Generalize to arbitrary dimensions.

(M,g) : smooth, orientable Riemannian spin^c mf.

$$\begin{tikzcd} T^*M \arrow[dr] & & \Lambda^\bullet M \arrow[dl] \\ & M & \end{tikzcd}$$

cotangent bundle

forms

$\Omega^\bullet(M) = \{$smooth sects. of $\Lambda^\bullet M\}$

differential forms

Module for $\mathcal{A} := C^\infty(M)$

$\mathcal{H} = L^2(\Omega^\bullet(M) \otimes \mathbb{C}, dvol_g)$

For $\xi \in \Omega^1(M) \otimes \mathbb{C}$, set

$X := g^{-1}\xi \in \Gamma(TM)$

vector fields

Creation- & annihilation ops

$\xi, \eta, \cdots \in \Omega^1(M) \otimes \mathbb{C}$, $\Phi \in \mathcal{H}$.

$$a^*(\xi)\Phi := \xi \wedge \Phi,$$

$$a(\xi)\Phi := \bar{X} \lrcorner \Phi.$$

Then

$$\{a^\#(\xi), a^\#(\eta)\} = 0,$$

CAR

$$\{a(\xi), a^*(\eta)\} = g^{-1}(\bar{\xi}, \eta),$$

$$a(\xi)\Omega^0 = 0, \quad a^*(\xi)\Omega^n = 0,$$

where $n = \dim M$.

$$\Gamma(\xi) := a^*(\xi) - a(\xi)$$

$$\bar{\Gamma}(\xi) := i(a^*(\xi) + a(\xi))$$

Then

$$\{\Gamma(\xi), \bar{\Gamma}(\eta)\} = 0$$

$$\{\overset{(-)}{\Gamma}(\xi), \overset{(-)}{\Gamma}(\eta)\} = -2\,\mathrm{Re}\, g^{-1}(\bar{\xi}, \eta)$$

two anticomm. sects. of Clifford bundle.

(M, g) spin^c $\Leftrightarrow \exists$ hermitian vector bundles $W, \bar{W}$, $\overset{(-)}{S} := \Gamma(\overset{(-)}{W})$, spinors, s.t.

dim M even:

$$\Omega^{\cdot}(M) \otimes \mathbb{C} \simeq \bar{S} \otimes_{A} S,$$

$$\Gamma(\xi) = 1 \otimes c(\xi), \quad \bar{\Gamma}(\xi) = \bar{c}(\xi) \otimes \gamma,$$

Clifford generators

$\xi \in \Omega^1(M)$.

dim M odd:

$$\Omega^{\cdot}(M)\otimes\mathbb{C} \simeq \bar{S} \otimes_A S \otimes \mathbb{C}^2$$

$$\Gamma(\xi) = 1\otimes c(\xi)\otimes\sigma^3,\ \bar{\Gamma}(\xi) = \bar{c}(\xi)\otimes 1\otimes\sigma^1$$

QM of Pauli electron and - positron on (M,g)

$$\mathcal{H}_e := L^2(S, dvol_g),\ \mathcal{H}_p := L^2(\bar{S}, dvol_g)$$

∇^S_A : spin^c connection on S
compatible with ∇_{LC}, i.e.,

em

$$\nabla^S_A(c(\xi)\psi) = c(\nabla_{LC}\xi)\psi + c(\xi)\nabla^S_A\psi$$

$\nabla^{\bar{S}}_A$: spin^c connection on $\bar{S}$
conjugate to ∇^S_A

Ambiguity of ∇^S_A :

$$(\nabla^S_{A_1} - \nabla^S_{A_2})\psi = i\alpha \otimes \psi, \ \alpha \in \Omega^1(M).$$

$$R_{\nabla^S_A}(X,Y) = \nabla^S_{A,X}\nabla^S_{A,Y} - \nabla^S_{A,Y}\nabla^S_{A,X} + \nabla^S_{A,[X,Y]}$$

$$F_A(X,Y) := 2^{-[\frac{n}{2}]} tr\left(R_{\nabla^S_A}(X,Y)\right)$$

em field strength

Pauli electron:

$D_A := c \circ \nabla^S_A$ on $\mathcal{H}_e$

Hamiltonian

$$H := \frac{1}{2m} D_A^2 + \phi$$

$$= \frac{1}{2m}\left(-\Delta + \frac{r}{4} + c(F_A)\right) + \phi$$

sa on $(\mathcal{D} \subseteq)\mathcal{H}_e$, $g=2$

"Observables":

$$\mathcal{A} = C^\infty(M) \quad \left(\hat{\mathcal{A}} = \psi DO(T^*M \times G_{n,2})\right)$$

"Pauli positron"

$$\bar{D}_A := \bar{c} \circ \nabla^{\bar{S}}_A \quad \text{on } \mathcal{H}_p$$

$$H = \frac{1}{2m} \bar{D}_A^{\,2} - \phi$$

<u>$N=1$ spectral data</u>

$$(\mathcal{H}, \mathcal{A}, D, (\gamma))$$

$$\mathcal{H} = \mathcal{H}_e, \mathcal{H}_p, \quad D = D_A, \bar{D}_A$$

"$\gamma = \gamma_5$": $\mathbb{Z}_2$ grading, n even,

$$\{\gamma, D\} = 0 = [\gamma, a], \quad a \in \mathcal{A}.$$

$e \leftrightarrow p$: "charge conjug."

<u>Theorem</u> (A.C.)

$(\mathcal{H}, \mathcal{A}, D, \gamma)$ encodes Riem. geometry of spin^c mf. (M, g) completely, (incl. de Rham th., ...).

Open problem: $\mathcal{A} \to \hat{\mathcal{A}}$?

QM of NR positronium

Positronium = groundstate of bound electron-positron

$$\Rightarrow \mathcal{H}_{e-p} = \mathcal{H}_p \otimes_{\mathcal{A}} \mathcal{H}_e \,(\otimes \mathbb{C}^2)$$

$$\simeq L^2(\Omega^{\cdot}(M) \otimes \mathbb{C}, dvol_g)$$

For $\Psi = \Psi_1 \otimes_A \Psi_2$, $\Psi_1 \in \mathcal{H}_p$, $\Psi_2 \in \mathcal{H}_e$,

$$\nabla \Psi := (\nabla_A^{\bar{S}} \Psi_1) \otimes \Psi_2 + \Psi_1 \otimes_A \nabla_A^S \Psi_2$$

$$= \nabla_{LC} \Psi \quad \text{indep. of } A$$

$$\mathcal{D} := \Gamma \circ \nabla, \quad \bar{\mathcal{D}} := \bar{\Gamma} \circ \nabla.$$

$$\{\mathcal{D}, \bar{\mathcal{D}}\} = 0, \quad \mathcal{D}^2 = \bar{\mathcal{D}}^2. \quad \#$$

$$H = \frac{1}{2\mu} \mathcal{D}^2 = \frac{1}{2\mu} \bar{\mathcal{D}}^2 \text{ on } \mathcal{H}_{e-p}$$

<u>$N = (1,1)$ spectral data</u>

$$\Big(\mathcal{H}_{e-p}, \mathcal{A}, \mathcal{D}, \bar{\mathcal{D}}, \underbrace{(\gamma, \bar{\gamma})}_{\text{"real structure"}}\Big)$$

Conventional interpretation:

$$d := \frac{1}{2}(\mathcal{D} - i\bar{\mathcal{D}}), \quad d^* = \frac{1}{2}(\mathcal{D} + i\bar{\mathcal{D}})$$

$$\# \Longleftrightarrow d^2 = (d^*)^2 = 0.$$

d: exterior derivative